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Construction of soliton cellular automaton from the vertex model—the discrete 2D Toda equation and the Bogoyavlensky lattice

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Abstract. We study the soliton cellular automaton (SCA) in (2+1)-dimensions from the viewpoint of the integrable vertex model. As in our previous paper, we relate the SCA, the so-called box–ball system, to an integrable vertex model associated with the Bogoyavlensky lattice. We extend this framework and introduce the (2 + 1)-dimensional SCA, which can be interpreted as the ultradiscretization of the 2D Toda equation. We also construct the N-soliton solutions for this system.

1. Introduction

The Bogoyavlensky lattice [1–5] is a differential-difference equation given by

$$\frac{\mathrm{d}\mathcal{V}_n}{\mathrm{d}t} = \mathcal{V}_n \sum_{k=1}^{M} (\mathcal{V}_{n+k} - \mathcal{V}_{n-k}) \tag{1.1}$$

where $\mathcal{V}_n \equiv \mathcal{V}_n(t)$ (for $t \in \mathbb{R}$, $n \in \mathbb{Z}$), and M is an arbitrary positive integer. It has been shown by using the inverse scattering method that this model is integrable in the Liouville sense not only for the classical case but also for the quantum case [3]. The N-soliton solution is also constructed by use of the bilinear equation [5]. The Hamiltonian structure of the Bogoyavlensky lattice is related to the lattice W algebra [4,6], and we can regard the hierarchy of the Bogoyavlensky lattice (a set of commutative flows including equation (1.1)) as the discrete analogue of the $(M + 1)$-reduced Kadomtsev–Petviashvili (KP) hierarchy [4].

Recent interest in the Bogoyavlensky lattice is due to the explicit relation with the soliton cellular automata (SCA). By use of the 'ultradiscretization' procedure [7], Tokihiro *et al* pointed out a relationship between the bilinear equation for the Bogoyavlensky lattice and the evolution equation for the SCA, the so-called 'the box–ball system' [8, 9], which is a generalization of the system introduced by Takahashi and Satsuma [10]. Since this bilinear equation is a reduction of the Hirota equation [11], the N-soliton solution for the box–ball system is constructed [8, 9] by making use of the Casorati determinant solution for the Hirota equation [12]. This ultradiscretization partly provides an answer to Wolfram's question: '*What is the correspondence between cellular automata and continuous systems?*' [13].

Recently, we proposed another new procedure called 'crystallization' to give the SCA from the Bogoyavlensky lattice in [14]. There, the quantized Lax matrix for the Bogoyavlensky lattice is regarded as the Boltzmann weight of the integrable vertex model on the two-dimensional square lattice. The configurations on each vertex cannot be generally fixed at a finite temperature, but they are uniquely determined at a zero temperature. We call this process the *crystallization*. A crucial fact which was clarified in [14] is that, in such a unique configuration, variables on the *vertical* edges exactly coincide with the time evolution of Takahashi–Satsuma's box–ball system. In this paper as a continuation of [14] we shall consider the integrable vertex model associated with the quantum Bogoyavlensky lattice, and study the cellular automaton on the *horizontal* edges in detail. Our main claim here is that, in the $M \to \infty$ limit, this cellular automaton can be interpreted as the ultradiscretized 2D Toda equation while the cellular automaton on the *vertical* edge is identified with Takahashi–Satsuma's box–ball system. We shall also construct the N-soliton solution for this (2 + 1)-dimensional system, and propose the reduction condition to obtain the soliton solution for a finite M case.

This paper is organized as follows. In section 2, we consider a relationship between the Bogoyavlensky lattice and the 2D Toda equation. We show that the bilinear equation for the 2D Toda equation is given when we modify that for the Bogoyavlensky lattice (1.1) and take a limit $M \to \infty$. We further time-discretize this bilinear equation, and construct the N-soliton solutions. Using the full-discretized bilinear equation, we construct *two* integrable difference equations as the discrete 2D Toda equation; one is essentially the same as the Bogoyavlensky lattice, the other is the dynamical system which will be shown in section 3 to be related to the integrable vertex model. In section 3, we introduce the 'crystallized' integrable vertex model on a two-dimensional square lattice, and consider the relationship between its unique configuration and the SCA. We first briefly review the results in [14], and define the box–ball system in terms of the two-dimensional integrable vertex model. Next we show that the dynamical system related to the *horizontal* edges coincides with the ultradiscretization of the 2D Toda equation in the limit $M \to \infty$ while variables on the *vertical* edges give the box–ball system. In section 4 we construct the N-soliton solution for the ultradiscretized 2D Toda equation, which gives a soliton solution of the SCA in (2 + 1) dimensions. We further study how to give a reduction condition to construct the N-soliton solutions for the box–ball system. The last section is devoted to the concluding remarks.

2. Transformation from the Bogoyavlensky lattice to the 2D Toda equation

We shall derive discrete analogues of the 2D Toda equation from the Bogoyavlensky lattice (1.1). We modify the dynamical variables for the Bogoyavlensky lattice as $\mathcal{V}_{Mn+j} \equiv \mathcal{V}_n^{(j)}$ (for $j = 0, 1, \ldots, M-1$) with a condition $\mathcal{V}_n^{(M)} = \mathcal{V}_{n+1}^{(0)}$, and rewrite the original equation (1.1) as

$$\frac{\mathrm{d}\mathcal{V}_n^{(j)}}{\mathrm{d}t} = \mathcal{V}_n^{(j)}\left(\sum_{k=0}^{j}\mathcal{V}_{n+1}^{(k)} + \sum_{k=j+1}^{M-1}\mathcal{V}_n^{(k)} - \sum_{k=0}^{j-1}\mathcal{V}_n^{(k)} - \sum_{k=j}^{M-1}\mathcal{V}_{n-1}^{(k)}\right). \tag{2.1}$$

We find that this equation can be written more simply as

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\mathcal{V}_n^{(j+1)}}{\mathcal{V}_n^{(j)}} = \mathcal{V}_{n+1}^{(j+1)} + \mathcal{V}_{n-1}^{(j)} - \mathcal{V}_n^{(j)} - \mathcal{V}_n^{(j+1)}. \tag{2.2}$$

We note that, in the limit $M \to \infty$, the Bogoyavlensky lattice (2.2) coincides with the equation of motion for the $\mathfrak{sl}_\infty$ Toda field theory [15–17], and that we can regard equation (2.2) as a (2 + 1)-dimensional equation by identifying the superscript j with another space dimension besides n [18]. Explicitly we set $\mathcal{V}_n^{(j)} \to \Delta\mathcal{V}_n(\Delta j, t)$ where Δ is a unit size of a new coordinate

j, and equation (2.2) reduces to the 2D Toda equation in the limit $\Delta \to 0$;

$$\frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}y}\log \mathcal{V}_n = \mathcal{V}_{n+1} - 2\mathcal{V}_n + \mathcal{V}_{n-1} \tag{2.3}$$

where we use notations $y = \Delta j$ and $\mathcal{V}_n \equiv \mathcal{V}_n(y, t)$. In this sense we can regard equation (2.2) as a semi-discrete 2D Toda equation.

To construct the soliton solution for the Bogoyavlensky lattice (2.2), we use the τ-function $\tau_n^{(j)}$ (for $n \in \mathbb{Z}$, $j = 0, 1, \ldots M-1$),

$$\mathcal{V}_n^{(j)} = \frac{\tau_{n+1}^{(j+1)}\tau_{n-1}^{(j)}}{\tau_n^{(j+1)}\tau_n^{(j)}}.$$

Here we suppose a condition, $\tau_n^{(M)} = \tau_{n+1}^{(0)}$. Then equation (2.2) reduces to the bilinear equation,

$$D_t\tau_n^{(j+1)} \cdot \tau_n^{(j)} = \tau_{n+1}^{(j+1)}\tau_{n-1}^{(j)} - \tau_n^{(j+1)}\tau_n^{(j)} \tag{2.4}$$

where the operator D_t denotes Hirota's bilinear operator [19],

$$(D_t)^n f \cdot g = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(t)g(t')|_{t=t'}.$$

It is straightforward to give the N-soliton solution from the bilinear equation (2.4) in the standard way, and we leave the fully discrete case until later.

Since equation (2.2) has a continuous variable t, it is necessary to introduce the fully discrete equation to relate the Bogoyavlensky lattice with the cellular automata. The discretization of the time coordinate t in equation (2.2) is realized by introducing the full-discretization of the bilinear equation (2.4) [5] as

$$\tau_n^{(j+1),t+1}\tau_n^{(j),t} - \delta\tau_{n+1}^{(j+1),t+1}\tau_{n-1}^{(j),t} = (1-\delta)\tau_n^{(j+1),t}\tau_n^{(j),t+1}. \tag{2.5}$$

Here we impose a condition $\tau_n^{(M),t} = \tau_{n+1}^{(0),t}$ and δ denotes the unit size of time-coordinate t. One recovers equation (2.4) by taking the $\delta \to 0$ limit in equation (2.5). By setting

$$\mathcal{V}_n^{(j),t} = \frac{\tau_{n+1}^{(j+1),t+1}\tau_{n-1}^{(j),t}}{\tau_n^{(j+1),t+1}\tau_n^{(j),t}}$$

with the τ-function in equation (2.5), one obtains the time-discretization of equation (2.2),

$$\frac{\mathcal{V}_n^{(j+1),t+1}\mathcal{V}_n^{(j),t}}{\mathcal{V}_n^{(j+1),t}\mathcal{V}_n^{(j),t+1}} = \frac{(1-\delta\,\mathcal{V}_n^{(j),t})(1-\delta\mathcal{V}_n^{(j+1),t+1})}{(1-\delta\mathcal{V}_{n-1}^{(j),t})(1-\delta\mathcal{V}_{n+1}^{(j+1),t+1})}. \tag{2.6}$$

We remark that, after the coordinate transformation, equation (2.6) coincides with the equation of motion for the discrete affine $A_M^{(1)}$ Toda field theory [20], and that a rank of the underlying Lie algebra plays a role of the space dimension.

We introduce another new discrete analogue of the 2D Toda equation by making use of the same τ-function in equation (2.5). When we set a variable $V_n^{(j),t}$ (for $n, t \in \mathbb{Z}$ and $j = 0, 1, \ldots, M-1$) as

$$V_n^{(j),t} = \frac{\tau_n^{(j),t-1}\tau_{n-2}^{(j),t-1}}{(\tau_{n-1}^{(j),t-1})^2} \tag{2.7}$$

we get the equation of motion for the variable $V_n^{(j),t}$,

$$\frac{V_n^{(j-1),t+1}V_n^{(j),t}}{V_{n-1}^{(j-1),t}V_n^{(j),t+1}} = \frac{\Psi_{n+1}^{(j),t}\Psi_{n-1}^{(j),t}}{(\Psi_n^{(j),t})^2} \tag{2.8}$$

where $\Psi_n^{(j),t}$ is given by

$$\Psi_n^{(j),t} = -\delta \prod_{n'=n+1}^{\infty} \frac{1}{V_{n'}^{(j),t+1}} + \prod_{n'=n}^{\infty} \frac{1}{V_{n'}^{(j-1),t}}. \tag{2.9}$$

We note that the variable $V_n^{(j),t}$ and the dynamical variable $\mathcal{V}_n^{(j),t}$ for the discrete Bogoyavlensky lattice (2.6) are related to each other as

$$V_{n+1}^{(j),t+1}\mathcal{V}_{n-1}^{(j-1),t-1} = V_n^{(j-1),t}\mathcal{V}_n^{(j-1),t-1} \tag{2.10}$$

and that the variable $V_n^{(j),t}$ is written in terms of $\mathcal{V}_n^{(j),t}$ as

$$V_n^{(j),t} = \prod_{k=2}^{\infty} \frac{\mathcal{V}_{n-k+1}^{(j-k),t-k}}{\mathcal{V}_{n-k}^{(j-k),t-k}}.$$

By expanding equation (2.8) by an infinitesimal δ (a unit of discrete time t), and picking up the first order of δ, we obtain the time-continuum limit ($\delta \to 0$) of equation (2.8) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \frac{V_n^{(j)}}{V_n^{(j-1)}} = \prod_{n'=n+1}^{\infty} \frac{V_{n'}^{(j-1)}}{V_{n'}^{(j)}} - 2\prod_{n'=n}^{\infty} \frac{V_{n'}^{(j-1)}}{V_{n'}^{(j)}} + \prod_{n'=n-1}^{\infty} \frac{V_{n'}^{(j-1)}}{V_{n'}^{(j)}} \tag{2.11}$$

where $V_n^{(j)} \equiv V_n^{(j)}(t)$. Especially in the $M = 1$ case, we have only one independent variable $V_n^{(0)} \equiv V_n$, and the mapping (2.10) reduces to $\mathcal{V}_n = V_{n+1}V_{n+2}$. The time-evolution equation for V_n becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} V_n = (V_n)^2(V_{n+1} - V_{n-1})$$

which is related to the discrete modified Korteweg–de Vries (KdV) equation [21]. On the other hand, in the limit of $M \to \infty$, the variable $V_n^{(j)}$ reduces to $\mathcal{V}_n$ in the continuum limit of j (we applied the same procedure as that used to derive equation (2.3) from (2.2)), and equation (2.11) naturally reduces to the 2D Toda equation (2.3). Therefore, we can regard equation (2.8) as another full-discretization of the 2D Toda equation (2.3). It should be noted that although the discrete 2D Toda equation and its ultradiscretization were studied in [22], their equation of motion is different from equations (2.6) and (2.8).

Before closing this section, we give the soliton solutions for the discrete 2D Toda equation (2.6). By applying the standard method to the bilinear equation (2.5) under the limit $M \to \infty$, the N-soliton solution is obtained as [23]

$$\tau_n^{(j),t} = \sum_{\mathcal{J}\subset\mathcal{I}} \left(\prod_{i,k\in\mathcal{J}, i<k} A(p_i, q_i; p_k, q_k) \right) \exp\left(\sum_{i\in\mathcal{J}} (c_i + \xi_n^{(j),t}(p_i, q_i)) \right). \tag{2.12}$$

Here parameters c_i ($i = 1, 2, \ldots, N$) are arbitrary constants, and p_i (resp. q_i) denotes a velocity in the n- (resp. j-) direction. The set $\mathcal{I}$ is $\mathcal{I} = \{1, \ldots, N\}$, and $\mathcal{J}$ is all subsets of $\mathcal{I}$. The function $\xi_n^{(j),t}(p, q)$ and the scattering matrix $A(p_i, q_i; p_k, q_k)$ are respectively defined as follows:

$$\xi_n^{(j),t}(p, q) = n \log p + j \log q + t \log w(p, q) \tag{2.13}$$

$$A(p_i, q_i; p_k, q_k) = \frac{(x(p_i, q_i) - x(p_k, q_k))(x(p_i^{-1}, q_i^{-1}) - x(p_k^{-1}, q_k^{-1}))}{(x(p_i, q_i) - x(p_k^{-1}, q_k^{-1}))(x(p_i^{-1}, q_i^{-1}) - x(p_k, q_k))} \tag{2.14}$$

where functions $w(p, q)$ and $x(p, q)$ are given by

$$w(p, q) = \frac{1 - \delta(1 + x(p^{-1}, q^{-1}))}{1 - \delta(1 + x(p, q))} \tag{2.15}$$

$$x(p, q) = \frac{q(p - 1)}{q - 1}. \tag{2.16}$$

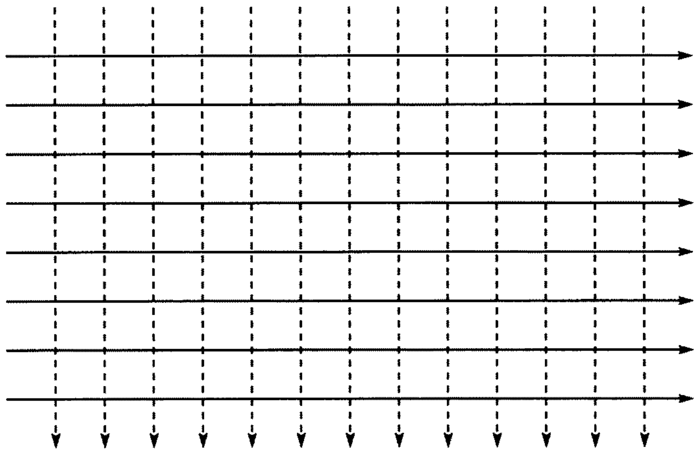


Figure 1. We assign variables on the edges of the two-dimensional square lattice. We classify the horizontal and vertical edges.

3. The vertex model and the SCA

We introduce the two-dimensional integrable vertex model associated with the Bogoyavlensky lattice following [14]. We show that the ultradiscretization of the 2D Toda equation also naturally appears in this vertex model.

3.1. The SCA as the Bogoyavlensky lattice

The SCA called 'the box–ball system' is defined by the following evolution equation [8]:

$$u_{N,j}^{T} = \min\left(\sum_{N'=-\infty}^{N-1} u_{N',j}^{T-1} - \sum_{N'=-\infty}^{N-1} u_{N',j}^{T}, 1 - \sum_{j'=1}^{j-1} u_{N,j'}^{T} - \sum_{j'=j}^{M} u_{N,j'}^{T-1}\right) \tag{3.1}$$

where M is a positive integer and is equal to M in the Bogoyavlensky lattice (1.1). This evolution equation describes how the balls move as time passes. The variable $u_{N,j}^{T}$ denotes the number of the ball $j\,(j = 1, \ldots, M)$ in the Nth box at the time T, and takes zero or one. We suppose an infinite lattice chain, and only one ball can occupy each box. As one can check easily from simple examples, the above box–ball system has soliton solutions [8, 24]. For our later convenience, we transform the evolution equation (3.1) into [8]

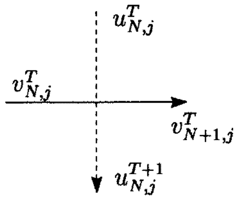
$$Y_N^{(j+1),T} + Y_{N-1}^{(j),T-1} = \max\left(Y_N^{(j),T} + Y_{N-1}^{(j+1),T-1}, Y_{N-1}^{(j+1),T} + Y_N^{(j),T-1} - 1\right) \tag{3.2}$$

where

$$u_{N,j}^{T} = Y_N^{(j),T} + Y_{N-1}^{(j+1),T} - Y_N^{(j+1),T} - Y_{N-1}^{(j),T}. \tag{3.3}$$

Here the ultradiscretized τ-function $Y_N^{(j),T}$ satisfies the quasi-periodic condition, $Y_N^{(M+1),T} = Y_N^{(1),T+1}$, and we have used the identity; $\min(A, B) = -\max(-A, -B)$.

In our previous paper [14], we established a novel relationship between the box–ball system (3.1) and the *crystallized* vertex model (i.e., the vertex model at zero temperature). In the framework of the vertex model, we consider the two-dimensional square lattice as shown in figure 1. The variables $u_{N,j}^{T} = \{0, 1\}$ for the box–ball system are on the *vertical* edges, and we assign the dynamical variables $v_{N,j}^{T} \in \mathbb{Z}_{\geqslant 0}$ on the *horizontal* edges. These variables evolve downwards and from the left to the right. For each vertex in the j(modulo M)th rows, we assign the dynamical variables $u_{N,j}^{T}$ and $v_{N,j}^{T}$ as

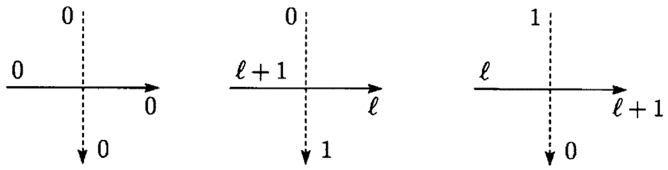


(3.4)

where variables obey a condition called the 'ice rule'; the sum of variables on arrows pointing into each vertex is equal to that on arrows coming out of the same vertex,

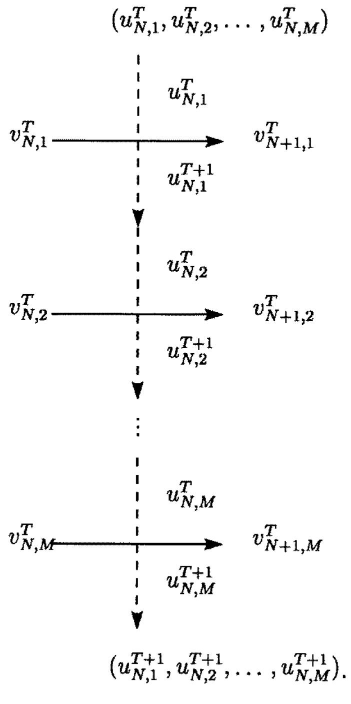
$$u_{N,j}^{T} + v_{N,j}^{T} = u_{N,j}^{T+1} + v_{N+1,j}^{T}. \tag{3.5}$$

To support the integrability of the box–ball system (3.1), we allow the following configuration for equation (3.4):



(3.6)

where $\ell \geqslant 0$. These are nothing but the configuration for the non-zero Boltzmann weight of the crystallized vertex model which originates from the quantized Lax matrix of the Bogoyavlensky lattice (see [14] for details). We note that a set of M-rows denote a unit of time step as follows:



Once the boundaries (top ends and left ends in figure 1) are given, a unique configuration of all vertices on two-dimensional lattices are determined by equation (3.6). The crucial fact is that

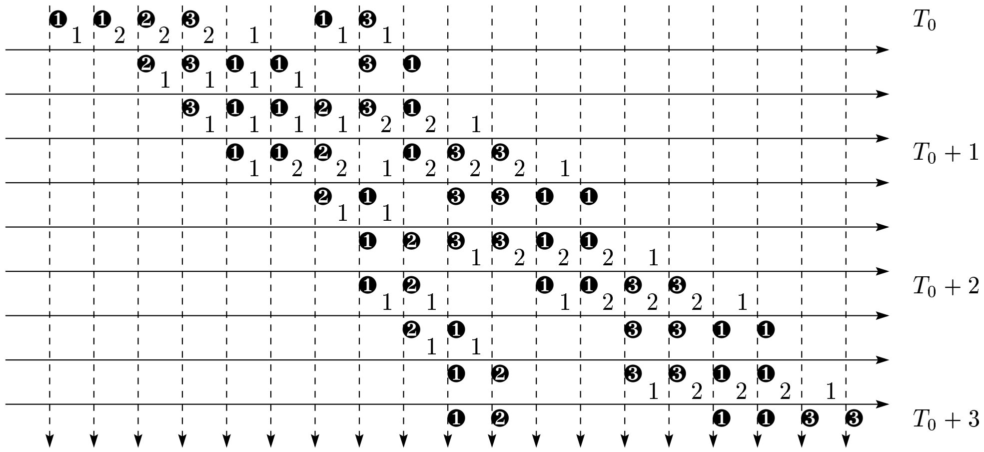


Figure 2. A two-soliton solution is given. Here ❶, ❷, ... denote $u_{N,1}^T = 1$, $u_{N,2}^T = 1$, ... respectively. Values of $v_{N,j}^T$ are explicitly written only when they are non-zero.

the evolution of the variables $u_{N,j}^T$ exactly coincides with that for the box–ball system (3.1). We give an example ($M = 3$) in figure 2, where the scattering of the two-soliton is shown. To distinguish variables on the vertical and the horizontal edges, we have respectively used ❶, ❷, ... for $u_{N,1}^T = 1$, $u_{N,2}^T = 1$, ... and 1, 2, 3, ... that denote the values of $v_{N,j}^T$. Note that we have omitted zero for simplicity.

The purpose of this paper is rather to consider the evolution equation related to the variables $v_{N,j}^T$. To this end, we define the variables $u_N^{(j),T}$ and $v_N^{(j),T}$ (for $j = 1, \ldots, M$) using variables $u_{N,j}^T$ and $v_{N,j}^T$ as

$$u_N^{(j),T} = \sum_{j'=1}^{j-1} u_{N,j'}^{T+1} + \sum_{j'=j}^{M} u_{N,j'}^T \tag{3.7a}$$

$$v_N^{(j),T} = \sum_{j'=j}^{M} v_{N,j'}^{T-1} + \sum_{j'=1}^{j-1} v_{N,j'}^T. \tag{3.7b}$$

Following the ice rule (3.5), these variables satisfy a relation,

$$v_N^{(j),T+1} + u_N^{(j),T} = v_{N+1}^{(j),T+1} + u_N^{(j),T+1}. \tag{3.8}$$

We transform the evolution equation (3.1) for $u_{N,j}^T$ into that for $v_N^{(j),T}$ using relations (3.7) and (3.8). We thus obtain

$$\begin{aligned} v_{N+1}^{(j),T+1} - v_N^{(j),T+1} = {} & \sum_{j'=1}^{j-1} v_N^{(j'),T+1} + \sum_{j'=j}^{M} v_N^{(j'),T} + \sum_{j'=1}^{j-1} \psi_N^{(j'),T} + \sum_{j'=j}^{M} \psi_N^{(j'),T-1} \\ & - \sum_{j'=1}^{j-1} \psi_N^{(j'),T+1} - \sum_{j'=j}^{M} \psi_N^{(j'),T} \end{aligned} \tag{3.9}$$

where

$$\psi_N^{(j),T} = \min\left(\sum_{T'=-\infty}^{T} v_N^{(j+1),T'}, 1 + \sum_{T'=-\infty}^{T} v_{N+1}^{(j),T'} \right).$$

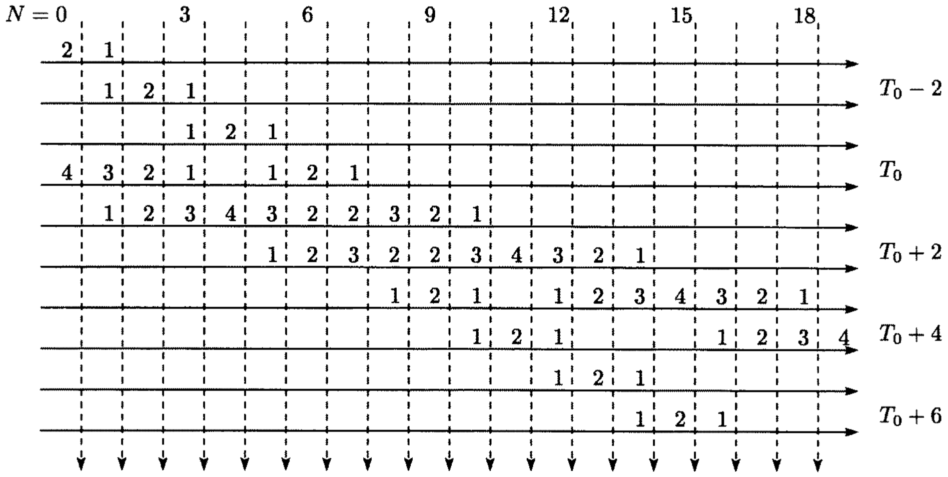


Figure 3. The evolution of $v_N^{(1),T}$ is given. This figure is given from the two-soliton solution in figure 2 by a transformation (3.7). We omitted zeros and $u_{N,j}^T$ here.

Here we have used a relation,

$$u_{N,j}^T = \psi_N^{(j),T-1} - \sum_{T'=-\infty}^{T-1} v_N^{(j),T'}.$$

The above equation (3.9) is the evolution equation for $v_N^{(j),t}$, which is associated with the variables on the horizontal edges in two-dimensional square lattice. We see that equation (3.9) is written in a simpler form,

$$v_N^{(j+1),T+1} - v_{N+1}^{(j+1),T+1} + v_{N+1}^{(j),T+1} - v_N^{(j),T} = \psi_N^{(j),T-1} - 2\psi_N^{(j),T} + \psi_N^{(j),T+1}. \tag{3.10}$$

The reason why we introduce the variable $v_n^{(j),t}$ by (3.7) may become clear from figure 3. Therein we give the evolution for $v_N^{(1),T}$, which corresponds to the two-soliton solution in figure 2. Here $1, 2, \ldots$ denote the values of $v_N^{(1),T}$, and we have omitted zeros and $u_{N,j}^T$. Note that unit time steps are different in those figures.

3.2. Ultradiscrete 2D Toda equation

When we rotate figure 3 and regard a coordinate N as a direction of the time, we notice that the soliton solution of $v_N^{(j),T}$ has a characteristic behaviour which reminds us of that for the ultradiscretized Toda equation [25]. In the rest of this section, we show that the evolution equation (3.10) coincides with the ultradiscrete 2D Toda equation.

To clarify the structures, we transform the coordinate of variables (N, T, j') of the previous section into (n, t, j) as

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} T \\ N \\ j' \end{pmatrix} = \begin{pmatrix} t \\ n \\ j \end{pmatrix}. \tag{3.11}$$

Hereafter we change sub- and superscripts of variables from $v_N^{(j'),T}$ into $v_n^{(j),t}$ and so on. By this transformation figure 3 changes into figure 4. As seen from an example in figure 4, the evolution for the variable $v_n^{(1),t}$ has the typical property of the soliton solutions:

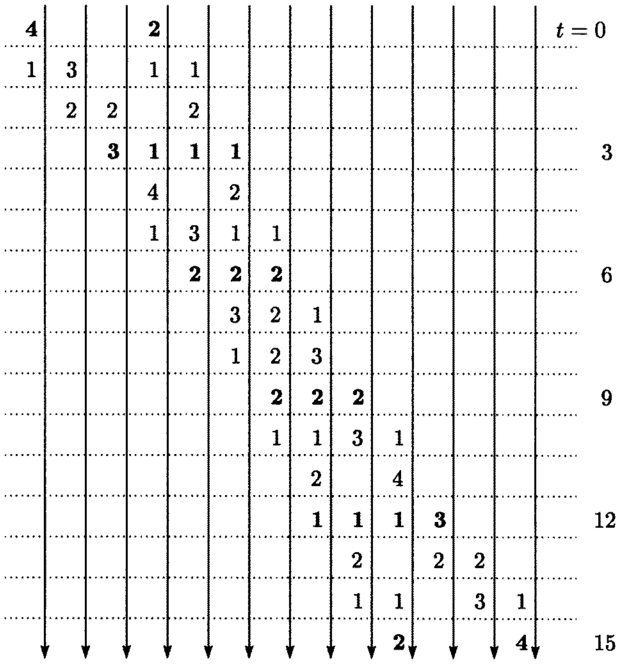


Figure 4. The evolution of $v_n^{(1),t}$ is given by the transformation (3.11) of figure 3.

- a soliton has the rapidity proportional to its amplitude,
- the number of solitons does not change after the collisions.

We shall later construct the soliton solutions explicitly by use of the τ-function in section 4.

The evolution equation (3.10) for the $v_n^{(j),t}$ is transformed by equation (3.11) into

$$v_{n-1}^{(j-1),t} - v_n^{(j-1),t+1} + v_n^{(j),t+1} - v_n^{(j),t} = \psi_{n+1}^{(j),t} - 2\psi_n^{(j),t} + \psi_{n-1}^{(j),t} \tag{3.12}$$

where $\psi_n^{(j),t}$ becomes

$$\psi_n^{(j),t} = \min\left(\sum_{n'=n}^{\infty} v_{n'}^{(j-1),t}, 1 + \sum_{n'=n+1}^{\infty} v_{n'}^{(j),t+1}\right).$$

One can check that the example in figure 4 evolves following equation (3.12). By comparing the evolution equation (3.12) with equation (2.8), we notice that equation (3.12) is the ultradiscretization of equation (2.8). Actually, when we substitute

$$V_n^{(j),t} = \exp\left(\frac{v_n^{(j),t}}{\varepsilon}\right) \qquad \Psi_n^{(j),t} = \exp\left(-\frac{\psi_n^{(j),t}}{\varepsilon}\right) \qquad \delta = -\exp\left(-\frac{1}{\varepsilon}\right)$$

into equation (2.8), it is straightforward to obtain equation (3.12) in a limit $\varepsilon \to 0$. Note that in the limit $M \to \infty$ we can forget about the quasi-periodicity for j and n in $v_n^{(j),t}$, and that we can regard $v_n^{(j),t}$ as variables defined on $(2+1)$-dimensions. In conclusion, the evolution equation for $v_n^{(j),t}$ in the limit $M \to \infty$ can be identified with the ultradiscrete 2D Toda equation (2.8). Important is that our two-dimensional integrable vertex model is related to two SCA; the variables $v_n^{(j),t}$ that originate from the variables on the horizontal edges of vertices indicate the discrete 2D Toda equation, while the variables on the vertical edges define the box–ball system.

The evolution equation (3.12) is bilinearized directly as follows. When we set the variables $v_n^{(j),t}$ as

$$v_n^{(j),t} = Y_n^{(j),t-1} - 2Y_{n-1}^{(j),t-1} + Y_{n-2}^{(j),t-1}$$

we see that $Y_n^{(j),t}$ satisfies

$$Y_n^{(j+1),t} + Y_n^{(j),t+1} = \max(Y_n^{(j+1),t+1} + Y_n^{(j),t}, Y_{n+1}^{(j+1),t+1} + Y_{n-1}^{(j),t} - 1). \tag{3.13}$$

This equation comes from equation (3.2) through a transformation (3.11). It is easy to see that the above equation is the ultradiscretization of the bilinear equation (2.5) when we set

$$\tau_n^{(j),t} = \exp\left(\frac{Y_n^{(j),t}}{\epsilon}\right) \qquad \delta = -\exp\left(-\frac{1}{\epsilon}\right). \tag{3.14}$$

4. The soliton solutions

4.1. Soliton solution of the ultradiscrete 2D Toda equation

We construct the N-soliton solution for the ultradiscrete 2D Toda equation (3.12) by applying the ultradiscretizing procedure to the N-soliton solution for the discrete 2D Toda equation given by (2.12). We consider the soliton solutions for equation (3.13), which is the ultradiscretization of the bilinear equation (2.5). One should substitute equations (3.14) and

$$\begin{aligned}
&A(p_i, q_i; p_k, q_k) = \exp\left(\frac{\mathcal{A}(P_i, Q_i; P_k, Q_k)}{\epsilon}\right) \\
&c_i = \frac{C_i}{\epsilon} \qquad \xi_{n,}^{(j),t}(p, q) = \frac{\Xi_n^{(j),t}(P, Q)}{\epsilon} \\
&w(p, q) = \exp\left(\frac{W(P, Q)}{\epsilon}\right) \qquad p = \exp\left(-\frac{P}{\epsilon}\right) \qquad q = \exp\left(-\frac{Q}{\epsilon}\right)
\end{aligned} \tag{4.1}$$

into equation (2.12), and take the limit $\epsilon \to 0$. We finally obtain the ultradiscretized τ-function as

$$Y_n^{(j),t} = \max_{\mathcal{J}\subset\mathcal{I}}\left(\sum_{i,k\in\mathcal{J},i<k} \mathcal{A}(P_i, Q_i; P_k, Q_k) + \sum_{i\in\mathcal{J}}(C_i + \Xi_n^{(j),t}(P_i, Q_i))\right) \tag{4.2}$$

where

$$\Xi_n^{(j),t}(P, Q) = -nP - jQ + W(P, Q)t \tag{4.3a}$$

$$\mathcal{A}(P_i, Q_i; P_k, Q_k) = -\min(|P_i + Q_i|, \ |P_k + Q_k|) \tag{4.3b}$$

$$W(P, Q) \equiv W(P) = \operatorname{sgn}(P)(|P| - 1). \tag{4.3c}$$

Note that the function $W(P, Q)$ does not depend on Q any more after the ultradiscretization. To describe the SCA, we assume that velocities P and Q are integers ($P, Q \in \mathbb{Z}$). For the simplicity of the scattering matrix $\mathcal{A}(P_i, Q_i; P_k, Q_k)$ (4.3b), we give the following conditions:

$$\begin{aligned}
&P_i Q_i \geqslant 1 \\
&\operatorname{sgn}(P_i) = \operatorname{sgn}(P_k) \qquad (P_i - P_k)(Q_i - Q_k) \geqslant 0 \qquad \text{for all } i, k.
\end{aligned} \tag{4.4}$$

We write explicitly the one- and two-soliton solutions:

- a one-soliton solution

$$Y_n^{(j),t} = \max(0\ , \ C + \Xi_n^{(j),t}(P, Q)) \tag{4.5}$$

- a two-soliton solution

$$Y_n^{(j),t} = \max(0, C_1 + \Xi_n^{(j),t}(P_1, Q_1), C_2 + \Xi_n^{(j),t}(P_2, Q_2),$$
$$C_1 + C_2 + \Xi_n^{(j),t}(P_1, Q_1) + \Xi_n^{(j),t}(P_2, Q_2) + \mathcal{A}(P_1, Q_1; P_2, Q_2)). \quad (4.6)$$

In figure 5, we give an example of the two-soliton solution for the ultradiscrete 2D Toda equation (3.12). Here we set $(P_1, Q_1; P_2, Q_2) = (4, 1; 2, 1)$ in equation (4.6). In this figure, the n- and j-coordinates are assigned to horizontal and vertical dotted lines respectively, and we write evolutions every *three* time steps. When we compare the emphasized numbers in figure 4 with that in figure 5, we find that the soliton solution for $v_n^{(1),t}$ in figure 4 is naturally embedded into the 2D Toda equation.

4.2. Reduction to the Bogoyavlensky lattice from the 2D Toda equation

As we have seen in section 2, the 2D Toda equation corresponds to the $M \to \infty$ limit of the Bogoyavlensky lattice. Therefore, to construct the soliton solution for the box–ball system which is associated with the Bogoyavlensky lattice (1.1), we must consider a proper reduction of the soliton solutions for the ultradiscrete 2D Toda equation (4.2).

Based on a relationship between the τ-functions (2.5) for the Bogoyavlensky lattice and the 2D Toda equation, we consider the reduction condition for the variable $Y_n^{(j),t}$ (3.13),

$$Y_n^{(M),t} = Y_{n+1}^{(0),t}. \quad (4.7)$$

This condition gives a constraint for the velocities P and Q of the the N-soliton solution (4.2). Instead of the velocity Q, we use sets of variables:

$$\{Q^{(j)}\} \qquad \text{for} \quad j = 0, 1, \ldots, M$$
$$\{\Delta Q^{(j)}\} \qquad \text{for} \quad j = 0, 1, \ldots, M-1$$

where $\Delta Q^{(j)} = Q^{(j)} - Q^{(j+1)}$. Roughly speaking, these variables play roles of the variable 'jQ' in equation (4.3*a*); $jQ \to Q^{(j)}$. Using the variables $Q^{(j)}$ and $\Delta Q^{(j)}$, the reduction condition (4.7) is written as

$$Q^{(0)} = Q^{(M)} + P$$

and the conditions (4.4) become

$$\begin{array}{lll} P_i, \Delta Q_i^{(j)} \in \mathbb{Z} & P_i \Delta Q_i^{(j)} \geqslant 1 & \\ \mathrm{sgn}(P_i) = \mathrm{sgn}(P_k) & (P_i - P_k)(\Delta Q_i^{(j)} - \Delta Q_k^{(j)}) \geqslant 0 & \text{for all } i, j, k. \end{array} \quad (4.8)$$

Therefore, when $P > 0$, the variables $Q^{(j)}$ should be chosen to satisfy

$$P = Q^{(0)} \geqslant Q^{(1)} \geqslant \cdots \geqslant Q^{(M-1)} \geqslant Q^{(M)} = 0. \quad (4.9)$$

With these variables $Q^{(j)}$ and $\Delta Q^{(j)}$, the soliton solutions (4.2) for the 2D Toda equation reduce to those for the box–ball system (3.1). We give the explicit forms of one- and two-soliton solutions as follows:

- a one-soliton solution

$$Y_n^{(j),t} = \max(0, C + \Xi_n^{(j),t}(P))$$

- a two-soliton solution

$$Y_n^{(j),t} = \max(0, C_1 + \Xi_n^{(j),t}(P_1), C_2 + \Xi_n^{(j),t}(P_2),$$
$$C_1 + C_2 + \Xi_n^{(j),t}(P_1) + \Xi_n^{(j),t}(P_2) + \mathcal{A}^{(j)}(P_1; P_2)).$$

$t = 0$

				4				2						
			1	3			1	1						
			2	2			2							
			3	1		1	1							
			4			**2**								
		1	3		1	1								
		2	2		2									
		3	1	1	1									
		4		2										
	1	3	1	1										
	2	2	2											
	$\underline{3}$	$\underline{2}$	$\underline{1}$											
$\underline{1}$	$\underline{2}$	$\underline{3}$												
2	2	2												
1	3	1												

$t = 3$

						3	1		1	1				
						4			2					
					1	3		1	1					
					2	2		2						
					3	**1**	**1**	**1**						
					4		2							
				1	3	1	1							
				2	2	2								
				$\underline{3}$	$\underline{2}$	$\underline{1}$								
			$\underline{1}$	$\underline{2}$	$\underline{3}$									
			2	2	2									
		1	1	3	1									
		2		4										
	1	1	1	3										
	2		2	2										

Figure 5. We give an example for the evolution of $v_n^{(j),t}$. The interaction between two solitons occurs where the underlined numbers such as $\underline{1}$, $\underline{2}$, ... exist. The emphasized numbers (**1**, **2**, ...) denote the evolution given in figure 4.

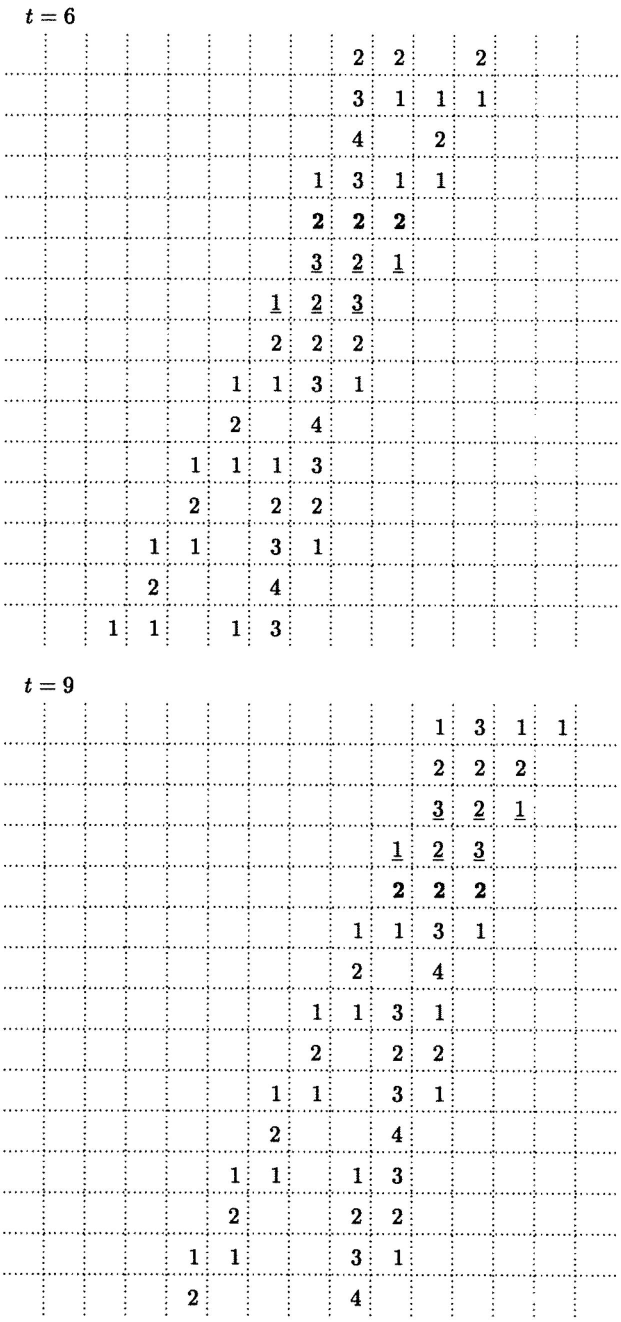


Figure 5. (Continued)

Here the function $\Xi_n^{(j),t}(P)$ and the scattering matrix $\mathcal{A}^{(j)}(P_1; P_2)$ are respectively given by

$$\Xi_n^{(j),t}(P) = -nP + Q^{(j)} + W(P)t \tag{4.10}$$

$$\mathcal{A}^{(j)}(P_1; P_2) = -\min(|P_1 + \Delta Q_1^{(j)}|, |P_2 + \Delta Q_2^{(j)}|) \tag{4.11}$$

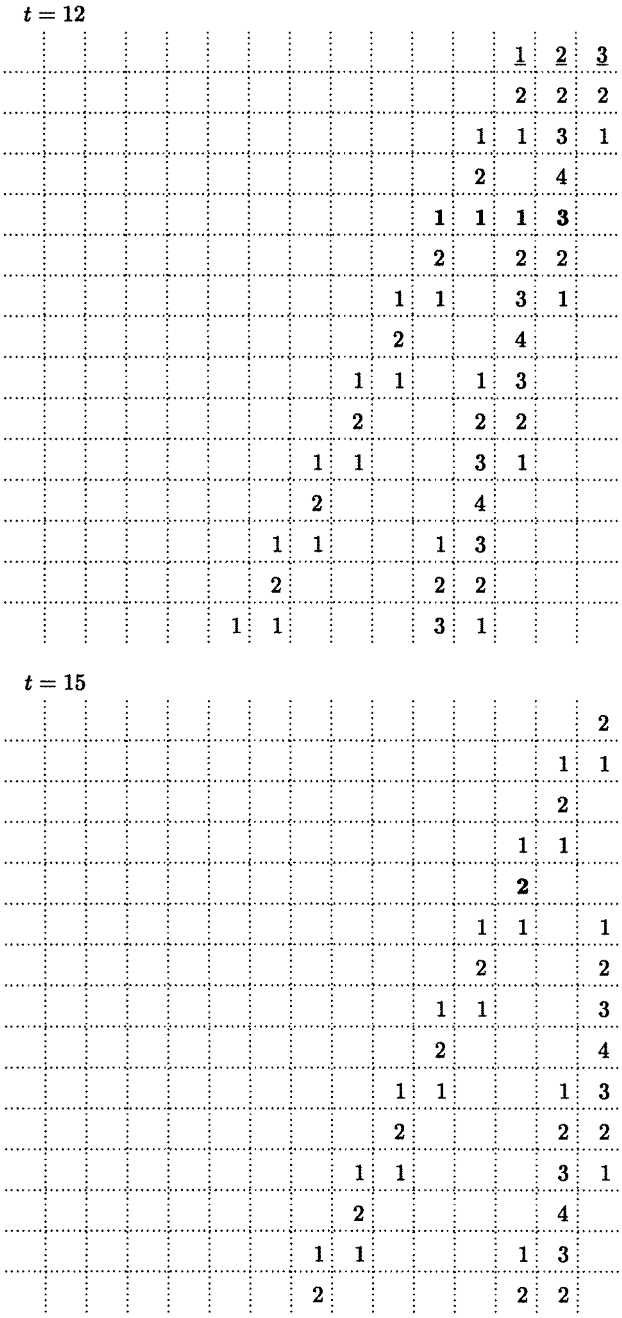


Figure 5. (Continued)

where we use the function $W(P)$ (4.3c).

In the $N \geqslant 3$ case, from a form of the two-soliton solution we suggest that the N-soliton solution is given by

$$Y_n^{(j),t} = \max_{\mathcal{J}\subset\mathcal{I}} \left(\sum_{s_i<s_k} \mathcal{A}^{(j)}(P_{s_i}; P_{s_k}) + \sum_{i\in\mathcal{J}} (C_{s_i} + \Xi_n^{(j),t}(P_{s_i})) \right)$$

where $\Xi_n^{(j),t}(P)$ is defined in equation (4.10), and $\mathcal{A}^{(j)}(P_i; P_k)$ is

$$\mathcal{A}^{(j)}(P_i; P_k) = -\min(|P_i + \Delta Q_i^{(j+k-i-1)}|, |P_k + \Delta Q_k^{(j+k-i-1)}|).$$

Here the variable $\Delta Q^{(j)}$ obeys $\Delta Q^{(j)} \equiv \Delta Q^{(j \bmod M)}$, $|\mathcal{J}|$ is the number of the elements of $\mathcal{J}$, and $s_i, s_k \in \{1, \ldots, |\mathcal{J}|\}$. See that the scattering matrix $\mathcal{A}^{(j)}(P_i; P_k)$ depends on the difference $(k-i)$. After the coordinate transformation (3.11), we can check that these soliton solutions coincide with that for the box–ball system in [8] where the N-soliton solution was constructed from the Casorati determinant solution for the Hirota equation [12].

5. Concluding remarks

We have considered two SCA associated with the crystallized vertex model (i.e., the vertex model at zero temperature) on the square lattice. This vertex model is defined based on the quantized Lax matrix of the Bogoyavlensky lattice. Our main claim in this paper is that the evolution of variables on the *horizontal* edges of vertices exactly coincides with the evolution equation for the ultradiscrete 2D Toda equation while the variables on the *vertical* edges follow the evolution equations of the box–ball system. Here as is often used, the rank of the underlying Lie algebra in the integrable system is treated as the second coordinate of the two-dimensional space [18, 20]. We have also constructed associated soliton solutions explicitly.

We list future problems:

- Are there other integrable dynamical equations to which we can apply both *ultradiscretization* and *crystallization*?
- How can we interpret the motion of the poles for soliton solutions [26] in the picture of ultradiscretization?

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